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Absolute Continuity in Noncommutative Measure Theory

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Abstract Recent results on absolute continuity of Banach space valued operators and convergence theorems on operator algebras are deepened and summarized. It is shown that absolute continuity of an operator T on a von Neumann algebra M with respect to a positive normal functional ψ on M is not implied by the fact that the null projections of ψ are the null projections of T. However, it is proved that the implication above is true whenever M is finite or T is weak*-continuous. Further it is shown that the absolute value preserves the Vitali-Hahn-Saks property if, and only if, the underlying algebra is finite. This result improves classical results on weak compactness of sets of noncommutative measures.

Keywords Von Neumann algebras · Absolute continuity of operators · Vitali-Hahn-Saks Theorem

1 Introduction and Preliminaries

In mathematical foundations of quantum theory laid down by J. von Neumann the probability and measure theory has to be extended to noncommutative structures given by operator algebras. One of the important topics in this research area which has received considerable attention recently is the study of noncommutative convergence theorems. The present paper is devoted to this line. In particular, the aim of this note is to deepen some recent results concerning the vector-valued noncommutative Vitali-Hahn-Saks Theorem. Our investigation is based on the ideas obtained jointly by E. Chetcuti and the author in the papers [4, 5].

The Vitali-Hahn-Saks Theorem is one of the basic principles of measure theory which can be viewed as a result on automatic uniform continuity of converging sequences of measures. In classical measure and probability theory the concept of absolute continuity is fundamental for convergence theorems and conditional expectation, respectively. The absolute continuity of scalar noncommutative measures has been treated in [2, 4]. In Sect. 2 of this

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work we continue this research by studying absolute continuity of general operators between von Neumann algebras and locally convex spaces. First we show that the absolute continuity of an infinite-dimensional operator T with respect to a normal positive functional ψ need not be equivalent to the statement that null projections of ψ are null projections of T. (It is of course true for finite-dimensional operators.) On the other hand, applying noncommutative Vitali-Hahn-Saks Theorem proved in [5] we show that this equivalence holds true for all finite von Neumann algebras. In Sect. 3 an interesting interplay of functional and its absolute value (variation) is treated. We shall prove that the transition to absolute values preserves the Vitali-Hahn-Saks property precisely for finite von Neumann algebras. This result was proved in [5] for σ-finite von Neumann algebras.

Let us now recall basic concepts and fix the notation. If X is a Banach space we shall denote by X_1 and X^* its closed unit ball and dual, respectively. Throughout the paper M shall stand for a von Neumann algebra. We denote by M_* and M_{*+} the predual and positive part of the predual of M, respectively. By $P(M)$ we shall denote the projection lattice of M, i.e. $P(M) = \{p \in M \mid p = p^* = p^2\}$. M is called σ-finite if each system of nonzero orthogonal projections in M is at most countable. A positive functional ψ on M is said to be faithful if $\psi(x) > 0$ whenever x is a positive nonzero element. If M acts on a Hilbert space H and $\xi, \eta \in H$, we shall use the following notation for normal vector functionals: $\omega_{\xi,\eta}(x) = (x\xi, \eta)$ $(x \in M)$; $\omega_\xi = \omega_{\xi,\xi}$. Each functional $\psi \in M_{*+}$ generates a seminorm $\|\cdot\|_\psi$ on M by $\|x\|_\psi = \sqrt{\psi(x^*x + xx^*)}$. The collection of all such seminorms determines the σ-strong* topology on M. A functional $\varphi \in M_*$ is said to be absolutely continuous with respect to a positive normal functional $\psi \in M_{*,+}$ (in symbols $\varphi \ll \psi$) if the restriction of φ to the unit ball M_1 is continuous at origin with respect to the seminorm $\|\cdot\|_\psi$. Let $K \subset M_*$. We say that K is pointwise absolutely continuous with respect to ψ (in symbols $K \ll_p \psi$) if each $\varphi \in K$ is absolutely continuous with respect to ψ. K is said to be uniformly absolutely continuous with respect to ψ (in symbols $K \ll_u \psi$) if the restriction of K to the unit ball M_1 is equicontinuous at zero with respect to the seminorm $\|\cdot\|_\psi$. In that case ψ is called the control functional for K. By famous Akemann's theorem a bounded set K in M_* is weakly relatively compact if, and only if, it admits a control functional (see [1]). We shall use the following notation for a set $K \subset M_*$ in the predual of M: $K_p = \{\psi \in M_{*,+} \mid K \ll_p \psi\}$ and $K_u = \{\psi \in M_{*,+} \mid K \ll_u \psi\}$. In the sequel, we shall often apply the following noncommutative Vitali-Hahn-Saks Theorem proved by E. Chetcuti and the author in [5].

Theorem 1.1 *A von Neumann algebra M is finite if, and only if, for any weakly relatively compact set $K \subset M_*$ the following equality holds*

$$\emptyset \neq K_p = K_u.$$

Let us now suppose that X is a locally convex space. An operator $T\colon M \to X$ is called completely additive if $T(\sum_\alpha p_\alpha) = \sum_\alpha T(p_\alpha)$ for any system (p_α) of pairwise orthogonal projections in M. (The sum on the right-hand side is supposed to converge in the topology of the space X.) By the symbol $B_{ca}(M, X)$ we shall denote the space of all continuous operators between M and X which are completely additive. In case when $X = \mathbb{C}$, the space $B_{ca}(M, \mathbb{C})$ reduces to the predual of M. From this point of view, $B_{ca}(M, X)$ can be seen as vector-valued predual of M.

2 Absolute Continuity of Operators

Definition 2.1 Let $T\colon M \to X$ be a continuous operator from a von Neumann algebra M into a locally convex space X with the topology τ. Suppose that ψ is a positive normal functional on a von Neumann algebra M. We say that T is absolutely continuous with respect to ψ (in symbols $T \ll \psi$) if the restriction of T to the unit ball M_1 is continuous at zero with respect to the seminorm $\|\cdot\|_\psi$. In other words, T is absolutely continuous with respect to ψ if $\lim_n Tx_n = 0$ whenever $(x_n) \subset M_1$ and $\lim_n \|x_n\|_\psi = 0$.

If $T \ll \psi$, then obviously

$$T(p) = 0 \text{ whenever } p \text{ is a projection in } M \text{ with } \psi(p) = 0. \tag{1}$$

It is well known in classical measure theory that condition (1) is in fact equivalent to absolute continuity. The same holds if T is a normal functional on a von Neumann algebra (see [2–4]). The following counterexample shows that the equivalence between absolute continuity and condition (1) breaks down if T is infinite dimensional. The construction follows that of [4] (Example 4.5).

Proposition 2.2 *Let H be a separable infinite-dimensional Hilbert space. There is a completely additive operator $T\colon B(H) \to c_0$ such that*

$$\operatorname{Ker} T \cap P(B(H)) = \operatorname{Ker} \psi \cap P(B(H))$$

and such that T is not absolutely continuous with respect to ψ. (c_0 denotes the space of all complex sequences with zero limit.)

Proof Let (e_n) be an orthonormal basis of H. Set

$$\psi = \sum_{n=2}^{\infty} \frac{1}{2^n} \omega_{e_n}.$$

Define an operator $T\colon B(H) \to c_0$ by

$$T(x) = (\omega_{e_1, e_{n+1}}(x))_{n=1}^{\infty}, \quad x \in B(H).$$

It is clear that T is bounded. Let us verify that T is completely additive. Indeed, let $p = \sum_{n=1}^{\infty} p_n$ where p, p_n's are projections in $B(H)$. Then

$$\left\| Tp - \sum_{n=1}^{N} Tp_n \right\|_{c_0} = \left\| \left(\left(p - \sum_{n=1}^{N} p_n \right) e_1, e_{k+1} \right)_k \right\|_{c_0} \le \left\| \left(p - \sum_{n=1}^{N} p_n \right) e_1 \right\| \to 0 \text{ as } N \to \infty.$$

Put

$$\xi_n = \frac{1}{\sqrt{2}}(e_1 + e_{n+1}), \quad n \in \mathbb{N},$$

and consider projection q_n onto one-dimensional space generated by ξ_n. It can be computed that

$$\omega_{e_1, e_{n+1}}(q_n) = \frac{1}{2} \quad \text{for all } n.$$

On the other hand,

$$\psi(q_n) = \frac{1}{2^{n+2}} \to 0 \quad \text{as } n \to \infty.$$

Hence,

$$\|T(q_n)\| \geq \frac{1}{2} \quad \text{for all } n$$

while $\psi(q_n) \to 0$ and so T is not absolutely continuous with respect to ψ. Let p denote projection onto span of e_1. Then it can be verified easily that

$$\operatorname{Ker} T \cap P(B(H)) = \operatorname{Ker} \psi \cap P(B(H)) = \{p\}. \qquad \square$$

Verifying condition (1) is much easier then verifying the absolute continuity itself. Therefore it is natural to ask under what circumstances (1) already implies absolute continuity. We show two positive results in this direction. Condition (ii) in the next theorem generalizes the result on absolute continuity of normal functionals proved in [4] (Proposition 2.2).

Theorem 2.3 *Let $T \in B_{ca}(M, X)$ where X is a locally convex space with the topology τ and let*

$$\operatorname{Ker} T \cap P(M) \supset \operatorname{Ker} \psi \cap P(M)$$

for a functional $\psi \in M_{*,+}$. Then $T << \psi$ provided that at least one of the following conditions is satisfied:

(i) *M is finite.*
(ii) *T is weak*–τ continuous.*

Proof Suppose that (i) is true. Let $\varrho(\cdot)$ be a continuous seminorm on X. Denote $X^*_{\varrho,1} = \{f \in X^* \mid |f(x)| \leq \varrho(x) \text{ for all } x \in X\}$. Consider the set $K_\varrho = \{f \circ T \mid f \in X^*_{\varrho,1}\} \subset M_*$. It is a bounded set. Assume that $p_n \searrow 0$ is a decreasing sequence of projections. Then for each $f \in X^*_{\varrho,1}$ we can estimate $|f(Tp_n)| \leq \varrho(Tp_n)$ and so the set K_ϱ is uniformly σ-additive. Akemann's theorem (see [1]) now states that K_ϱ is weakly relatively compact. As $K_\varrho \ll_p \psi$ we can use Theorem 1.1 and derive that $K_\varrho \ll_u \psi$. By the Hahn-Banach Theorem applied to any continuous seminorm on X we see that $T \ll \psi$.

Suppose (ii). We shall prove that if T is not absolutely continuous with respect to ψ, then $\operatorname{Ker} T \cap P(M) \neq \operatorname{Ker} \psi \cap P(M)$. In that case there is a sequence $(x_n) \subset M_1$ such that $\|x_n\|_\psi \to 0$ and $Tx_n \notin U$ for all n, where U is some nonzero open neighbourhood of 0 in X. Let

$$x_n = x_{n,1} - x_{n,2} + i(x_{n,3} - x_{n,4})$$

be a decomposition of the real and imaginary part of x_n into a difference of two orthogonal positive elements. Let x_i $(i = 1, 2, 3, 4)$ be an accumulation point of the set $\{x_{n,i} \mid n \in \mathbb{N}\}$ in the weak* topology. Set $x = x_1 - x_2 + i(x_3 - x_4)$. By the weak* continuity of T, Tx is an accumulation point of the set $\{Tx_n \mid n \in \mathbb{N}\}$. Therefore $Tx \notin U$ and so Tx is not zero. Observe that

$$\|x_n\|^2_\psi = 2[\psi(x^2_{n,1}) + \psi(x^2_{n,2}) + \psi(x^2_{n,3}) + \psi(x^2_{n,4})].$$

Hence $\lim_n \psi(x^2_{n,i}) = 0$ and using Cauchy-Schwarz inequality we obtain for $i = 1, \ldots, 4$

$$\psi(x_{n,i})^2 \leq \psi(1)^2 \psi(x^2_{n,i}) \to 0 \text{ as } n \to \infty. \tag{2}$$

Since $Tx \neq 0$ we can suppose without loss of generality that $Tx_1 \neq 0$. Let us write x_1 as a σ-convex combination of the projections $p_n's$:

$$x_1 = \sum_n \frac{1}{2^n} p_n.$$

Then $Tx_1 = \sum_n \frac{1}{2^n} Tp_n$ and so there is a projection p from the set $\{p_n \mid n \in \mathbb{N}\}$ such that $Tp \neq 0$. However, $\psi(x_1) = 0$ by (2) and consequently $\psi(p_n) = 0$ for all n. It means that $p \in \operatorname{Ker} \psi \setminus \operatorname{Ker} T$. □

3 Variations of Measures and Convergence Theorems

The results in this part demonstrate a difference between nonnegative and signed measures in noncommutative setting. In classical measure theory the properties of measures often pass to their variations. For instance, if μ is a signed measure that is absolutely continuous with respect to some probability measure ω, then the variation, $|\mu|$, is also absolutely continuous with respect to ω. Further, if K is a set of classical measures which is relatively compact in the convergence topology, then the set $|K|$ consisting of variations of measures from K is also relatively compact. We shall show that it is far from being true for noncommutative algebras. In noncommutative measure theory the variations of measures behave differently. It follows first of all from Theorem 1.1 saying that the Vitali-Hahn-Saks Theorem for signed measures does not hold for infinite von Neumann algebras. Before this result it had been known for a long time that the Vitali-Hahn-Saks Theorem holds for all positive measures on von Neumann algebras.

We recall a few facts on the polar decomposition of a normal functional. Let φ be a normal functional on M. Then there is a unique positive normal functional ω with the support $s(\omega)$ and unique partial isometry $v \in M$ such that

$$\varphi(x) = \omega(xv) \text{ and } v^*v = s(\omega).$$

The positive functional ω is denoted by $|\varphi|$ and is called the absolute value of φ. In commutative case this concept gives the usual variation of a complex measure. For $K \subset M_*$ we shall set $|K| = \{|\varphi| \mid \varphi \in K\}$. It was observed in [6] that K is weakly relatively compact provided that $|K|$ is weakly relatively compact. The main result due to Saitô proved in [6] says that a von Neumann algebra is finite if, and only if, for any weakly relatively compact set K in M_* the set $|K|$ is also weakly relatively compact. In other words, the transition to absolute value preserves compactness if, and only if, the underlying algebra is finite. As it is clear from Theorem 1.1, the Vitali-Hahn-Saks Theorem does not hold for all weakly relatively compact sets. It motivates the following definition.

Definition 3.1 $K \subset M_*$ is said to have the Vitali-Hahn-Saks property if

$$\emptyset \neq K_p = K_u.$$

The set with the Vitali-Hahn-Saks property appears to be weakly relatively compact. Indeed, suppose that $K \subset M_*$ has the Vitali-Hahn-Saks property. Then there is a control functional for K. By famous Akemann's compactness criterion K must be weakly relatively compact. (Akemann's theorem says that a bounded set $K \subset M_*$ is weakly relatively compact if, and only if, $K_u \neq \emptyset$.) On the other hand, by our result M is finite precisely when all weakly relatively compact subsets of M_* enjoy the Vitali-Hahn-Saks property. Combining this with the result of Saitô we see that for finite algebras the following holds: $K \subset M_*$ has the Vitali-Hahn-Saks property if, and only if, $|K|$ has the Vitali-Hahn-Saks property. Surprisingly enough, both implications in the previous equivalence characterize the finiteness of algebras. The following theorem was proved by E.Chetcuti and the author in [5] (Theorem 2.6) for σ-finite algebras.

Theorem 3.2 *Let M be a von Neumann algebra. The following statements are equivalent.*

(i) *M is finite.*
(ii) *$|K|$ has the Vitali-Hahn-Saks property implies that K has the Vitali-Hahn-Saks property, for all bounded $K \subset M_*$.*
(iii) *K has the Vitali-Hahn-Saks property implies that $|K|$ has the Vitali-Hahn-Saks property, for all bounded $K \subset M_*$.*

Proof By inspection of the proof of Theorem 2.6 in [5] one can find out that the σ-finiteness is not essential for most of the arguments. Only the implication (iii) implies (i) needs some modification. The proof of this implication in general case amounts to showing that for any properly infinite algebra M there is a bounded set $K \subset M_*$ with the Vitali-Hahn-Saks property such that $|K|$ does not have the Vitali-Hahn-Saks property. For this it is sufficient to construct K with the Vitali-Hahn-Saks property such that $|K|$ is not weakly relatively compact. Suppose that M is properly infinite. There is a sequence (e_n) of equivalent orthogonal projections in M with sum $\sum_n e_n = 1$. Let us choose a nonzero σ-finite projection $f_1 \leq e_1$. Then there are projections $f_i \leq e_i$ equivalent to f_1. Put $p = \sum_n f_n$. The projection p is nonzero properly infinite and σ-finite. Set $N = pMp$. By our previous construction in [5] (Theorem 2.6) there is a bounded separating set $L \subset N_*$ which has the Vitali-Hahn-Saks property and such that $|L|$ is not weakly relatively compact. Consider the set

$$K = \{\varphi(p \cdot p) \mid \varphi \in L\} \subset M_*.$$

It is clear that K is weakly relatively compact. Let us show that K has the Vitali-Hahn-Saks property. N is σ-finite and so there is a faithful normal state ω on N. The state $\omega(p \cdot p)$ is in K_p and so $K_p \neq 0$. Take $\psi \in K_p$. The restriction φ of ψ to N is faithful and so $\varphi \in L_p$. By the Vitali-Hahn-Saks property of L, $\varphi \in L_u$. Therefore $\psi(p \cdot p)$ is a control functional for K. Let q be a projection in M with $\psi(q) = 0$. Then $\varrho(pqp) = 0$ for all $\varrho \in L$. Consequently $pqp = 0$ because L is separating. Therefore $\psi(p \cdot p)$ is absolutely continuous with respect to ψ, giving that ψ is a control functional for K. Finally, it is easy to verify that for each ϱ in K which restricts to ϱ' on N we have $|\varrho| = |\varrho'|(p \cdot p)$. In other words, each $|\varrho|$ restricts to $|\varrho'|$ on N. As $|L|$ is not weakly relatively compact we conclude that $|K|$ cannot be weakly relatively compact as well. □

The arguments in the proof of the previous theorem strengthen the construction of Saitô producing a weakly relatively compact set K in the predual of any infinite algebra such that $|K|$ is not weakly relatively compact. We have shown that K can even be chosen to satisfy the Vitali-Hahn-Saks property. There are many characterizations of finiteness of von Neumann algebras of various kinds. Lattice theoretic characterization says that M is finite if, and only if, $P(M)$ is a modular lattice. Topological characterisation tells us that M is finite if, and only if, the $*$-operation is σ-strong* continuous. Our result may be viewed as new measure theoretic characterisation of finite algebras in terms of measures.

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